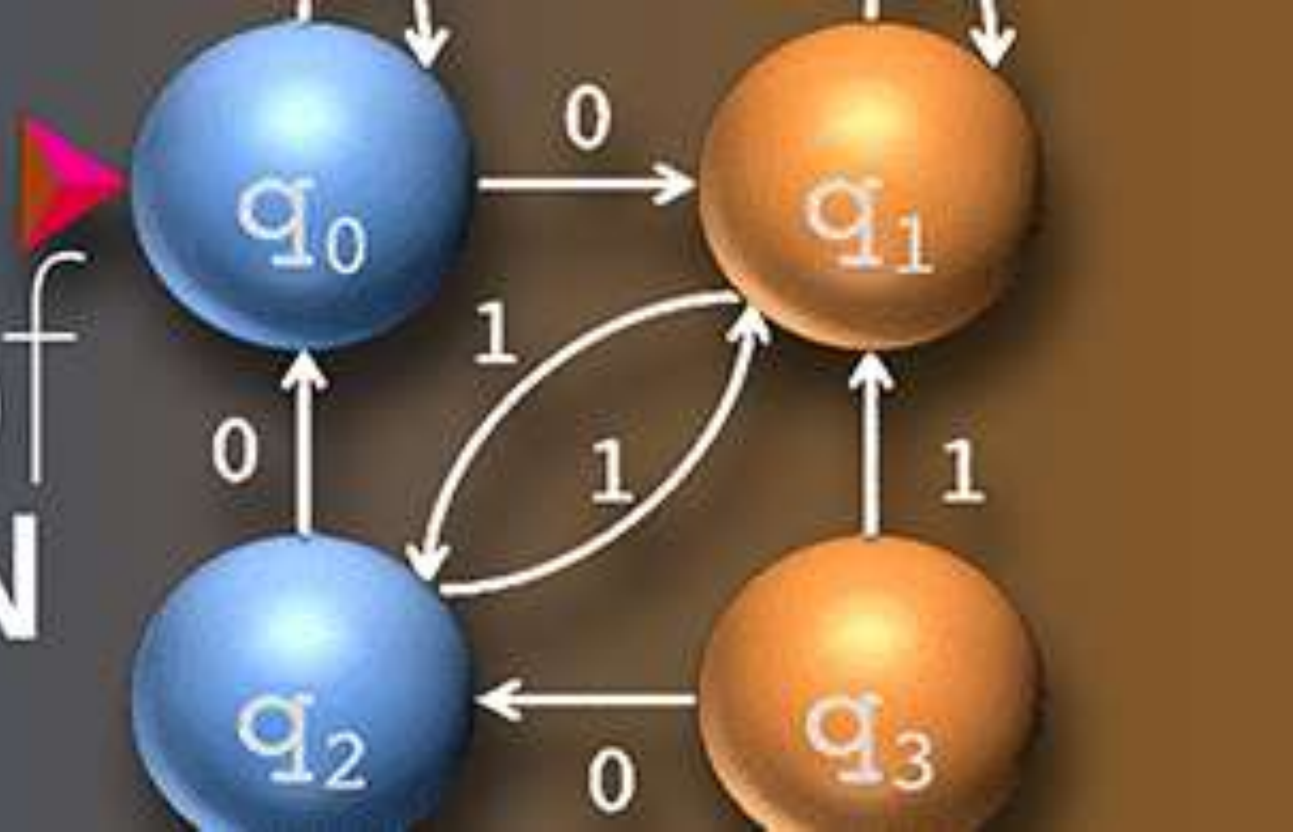
CSE 305

Theory of COMPUTATION



Lecture 5

Mathematical Preliminaries (1)

Md. Mijanur Rahman, Prof. Dr.

Dept. of Computer Science and Engineering, Jatiya Kabi Kazi Nazrul Islam University, Bangladesh.

www.mijanrahman.com

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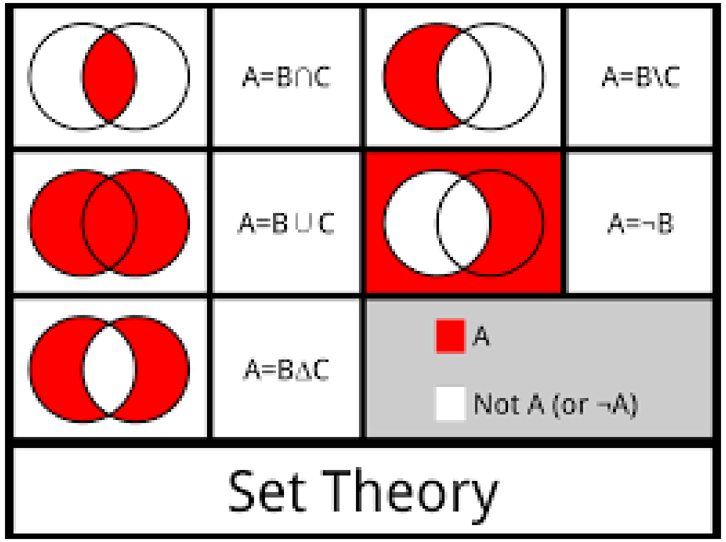
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Set Theory


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Set Theory

- **Set theory, branch of mathematics that deals with the properties of well-defined collections of objects, which may or may not be of a mathematical nature, such as numbers or functions.**
- Between the years 1874 and 1897, the German mathematician and logician Georg Cantor created a theory of abstract sets of entities and made it into a mathematical discipline. A set, wrote Cantor, is a collection of definite, distinguishable objects of perception or thought conceived as a whole. The objects are called elements or members of the set.
- **In naive set theory, a set is a collection of objects (called members or elements) that is regarded as being a single object. To indicate that an object x is a member of a set A one writes $x \in A$, while $x \notin A$ indicates that x is not a member of A.**
- **A set may be defined by a membership rule (formula) or by listing its members within braces. For example, S = {0, 2, 4, 6, 8, 10}, and A = {y|y is an English alphabet}.**
- The symbols $\in$ and $\notin$ denote set membership and non-membership. We can write $2 \in S$, and $3 \notin S$.

Types of Sets

- The sets are further categorized into different types, based on elements or types of elements. These different types of sets in basic set theory are:

- **Finite and Infinite Sets:** If a set contain a finite number of elements, it is said to be a finite set, else it is called an infinite set. An ***infinite set*** contains infinitely many elements. We cannot write a list of all the elements of an infinite set, so we sometimes use the ". . ." notation to mean "continue the sequence forever." Thus, we write the set of ***natural numbers*** N as {1, 2, 3, …}

- **Disjoint Sets:** Two sets A and B are said to be disjoint, if they do not have any element in common. For example, A = {x|x is a vowel}, and B = {y|y is a consonant},

- **Singleton Set:** A set with one member is sometimes called a ***singleton set.*** For example, B = {2},

- **Empty Set:** A set that contains no elements, { }, is called the **empty set** or ***null set*** and and is notated ∅.

Types of Sets

- **Subset and Superset:** For two sets A and B, we say that A is a ***subset*** of B, written A $\subseteq$ B, if every member of A also is a member of B. B is called the superset of A. We also say that ‘B contains A’ or ‘A is contained in B’. We say that A is a ***proper subset*** of B, if A is a subset of B and not equal to B. For example, A = {x|x is a vowel}, and B = {y|y is an English alphabet}.

- A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

- **Equal set:** Two sets are equal if they have same elements

- A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in *A*.

- **Power set:** A set of every possible subset.

- **Equivalent set:** Two sets are equivalent if they have same number of elements.

- **Subset:** When all the elements of set A belong to set B, then A is subset of B.

Set Theory Symbols

- There are several symbols that are adopted for common sets. They are given in the table:

Symbol	Corresponding Set
N	Represents the set of all Natural numbers i.e. all the positive integers. This can also be represented by Z^+ . Examples: 9, 13, 906, 607, etc.
Z	Represents the set of all integers The symbol is derived from the German word *Zahl*, which means number. Positive and negative integers are denoted by Z^+ and Z^- respectively. Examples: -12, 0, 23045, etc.
Q	Represents the set of Rational numbers The symbol is derived from the word *Quotient*. It is defined as the quotient of two integers (with non-zero denominator) Positive and negative rational numbers are denoted by Q^+ and Q^- respectively. Examples: $\frac{13}{9}, -\frac{6}{7}, \frac{14}{3}$, etc.
R	Represents the Real numbers i.e. all the numbers located on the number line. Positive and negative real numbers are denoted by R^+ and R^- respectively. Examples: $4.3, \pi, 4\sqrt{3}$,, etc.
C	Represents the set of Complex numbers. Examples: 4 + 3i, i, etc.

Set Theory Symbols

- Other Notations

Symbol	Symbol Name
{ }	set
A ∪ B	A union B
A ∩ B	A intersection B
A ⊆ B	A is subset of B
A ⊄ B	A is not subset B
A ⊂ B	proper subset / strict subset
A ⊃ B	proper superset / strict superset
A ⊇ B	superset
A ⊅ B	not superset
Ø	empty set
P (C)	power set
A = B	Equal set
Ac	Complement of A
a∈B	a element of B
x∉A	x not element of A

Set Theory Formulas

- n(A ∪ B) = n(A) +n(B) − n (A ∪ B)
- n(A∪B)=n(A)+n(B) {when A and B are disjoint sets}
- n(U)=n(A)+n(B)−n(A∩B)+n((A∪B)c)
- n(A∪B)=n(A−B)+n(B−A)+n(A∩B)
- n(A−B)=n(A∩B)−n(B)
- n(A−B)=n(A)−n(A∩B)
- n(Ac)=n(U)−n(A)
- n(PUQUR)=n(P)+n(Q)+n(R)−n(P∩Q)−n(Q∩R)−n(R∩P)+n(P∩Q∩R)

Operations on Set Theory

- **Operations on sets:** The four important set operations that are widely used are: Union of sets, Intersection of sets, Complement of sets, and Difference of sets

- **Union of two or more sets** is the set containing all the elements of the given sets. Union of sets can be written using the symbol "U". Suppose the union of two sets X and Y can be represented as X U Y. The union of two sets X and Y is equal to the set of elements that are present in set X, in set Y, or in both the sets X and Y. This operation can be represented as; **$X \cup Y = \{a: a \in X \text{ or } a \in Y\}$**

- **The intersection of two sets** A and B which are subsets of the universal set U, is the set which consists of all those elements which are common to both A and B. It is denoted by '∩' symbol. All those elements which belong to both A and B represent the intersection of A and B. Thus we can say that, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

- The **complement** of a set *A* contains everything that is *not* in the set *A*. The complement is notated *A'*, or A^c, *or sometimes ~A.*

- **Difference of two sets** A and B is the set of elements which are present in A but not in B. It is denoted as A-B.

Algebra of Sets

- Sets under the operations of union, intersection, and complement satisfy various laws (identities) which are listed in the Table.

Idempotent Laws	(a) $A \cup A = A$	(b) $A \cap A = A$
Associative Laws	(a) $(A \cup B) \cup C = A \cup (B \cup C)$	(b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	(a) $A \cup B = B \cup A$	(b) $A \cap B = B \cap A$
Distributive Laws	(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
De Morgan's Laws	(a) $(A \cup B)^c = A^c \cap B^c$	(b) $(A \cap B)^c = A^c \cup B^c$
Identity Laws	(a) $A \cup \emptyset = A$ (b) $A \cup U = U$	(c) $A \cap U = A$ (d) $A \cap \emptyset = \emptyset$
Complement Laws	(a) $A \cup A^c = U$ (b) $A \cap A^c = \emptyset$	(c) $U^c = \emptyset$ (d) $\emptyset^c = U$
Involution Law	(a) $(A^c)^c = A$	

Cartesian Product of Sets

- **What is an Ordered Pair?**
- An ordered pair is a pair of objects where one element is assigned first, and the other element is assigned second, denoted by (a,b). Here ‘a’ is called the first component, and ‘b’ is called the second component of the ordered set.
- **Cartesian Product of Sets:**
- A cartesian product of two non-empty sets A and B is the set of all possible ordered pairs where the first component of the pair is from A, and the second component of the pair is from B. The set of ordered pairs thus obtained is denoted by **A×B**.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

- ***Example:***

Let $A = \{1, 2\}$ *and* $B = \{4, 5, 6\}$

$$A \times B = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6)\}$$

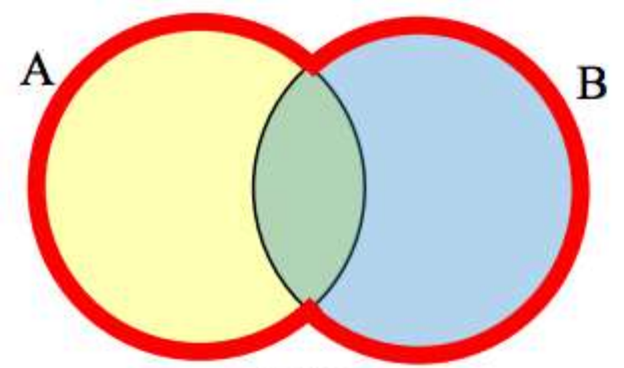
Cartesian Product of Sets

- **Properties of Cartesian Product:**

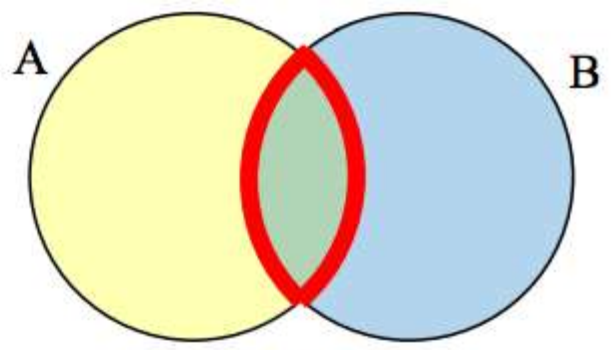
  1. The Cartesian Product is non-commutative: $A \times B \neq B \times A$
  2. $A \times B = B \times A$, only if $A = B$
  3. The cardinality of the Cartesian Product is defined as the number of elements in $A \times B$ and is equal to the product of cardinality of both sets: $|A \times B| = |A| * |B|$
  4. $A \times B = \{\emptyset\}$, if either $A = \{\emptyset\}$ or $B = \{\emptyset\}$

Venn Diagram

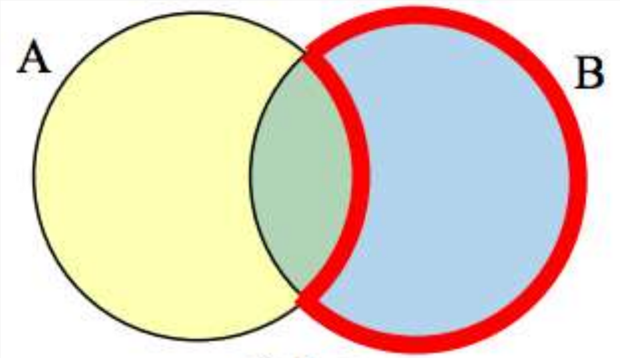
- A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.



$A \cup B$



$A \cap B$



$A^c \cap B$

Set Cardinality

- **Cardinality:** Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set. The number of elements in a set is the cardinality of that set.

- The cardinality of the set A is often notated as $|A|$ or $n(A)$

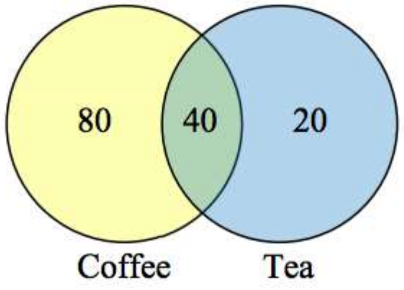
- **CARDINALITY PROPERTIES**

  1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

  2. $n(Ac) = n(U) - n(A)$

Example

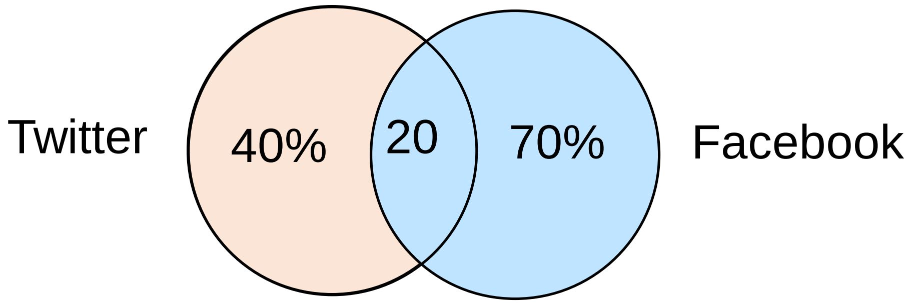
- A survey asks 200 people “What beverage do you drink in the morning”, and offers choices:
  - Tea only
  - Coffee only
  - Both coffee and tea



- Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?
- **Answers:**
  - This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.
  - We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.
  - 200 – 20 – 80 – 40 = 60 people who drink neither.

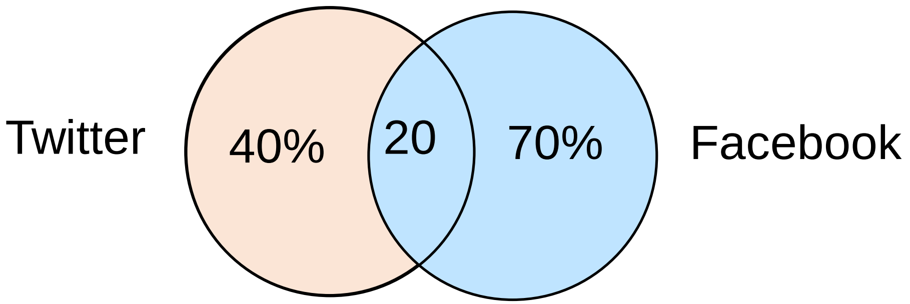
Example

- A survey asks: “Which online services have you used in the last month?”
  - Twitter
  - Facebook
  - Have used both
- The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?



How many people have used neither Twitter or Facebook?

Example

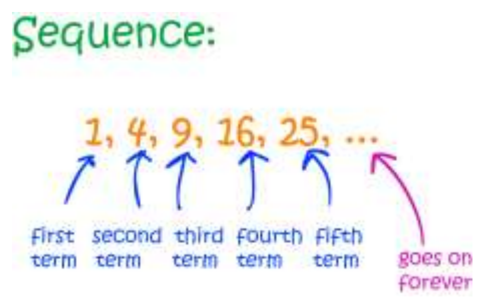


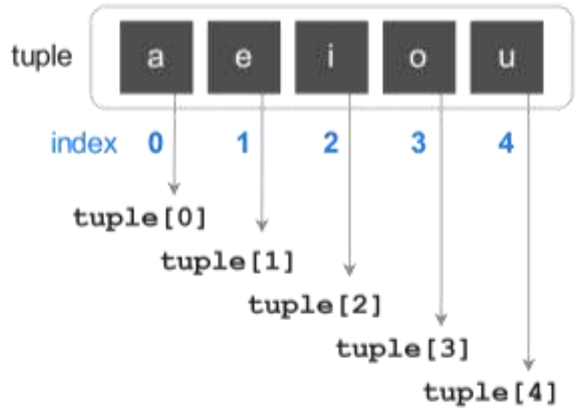
- **Answers:**
  - *T* be the set of all people who have used Twitter,
  - *F* be the set of all people who have used Facebook.
- While the cardinality of *F* is 70% and the cardinality of *T* is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice.
- To find the cardinality of $F \cup T$, we can add the cardinality of *F* and the cardinality of *T*, then subtract those in intersection that we've counted twice.

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$
$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

- To find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. Since the universal set contains 100% of people and the cardinality of $F \cup T$ = 90%, the cardinality of $(F \cup T)^c$ must be the other 10%.

Sequences and Tuples





Sequences and Tuples

- **A sequence of objects is a list of objects in some order.** We usually designate a sequence by writing the list within parentheses.
- For example, the sequence 7, 21, 57 would be written as (7, 21, 57).
- **The order doesn't matter in a set, but in a sequence it does**. Hence (7, 21, 57) is not the same as (57, 7, 21).
- **Similarly, repetition does matter in a sequence, but it doesn't matter in a set.**
- Thus (7, 7, 21, 57) is different from both of the other sequences, whereas the set {7, 21, 57} is identical to the set {7, 7, 21, 57}.

Sequences and Tuples

- As with sets, sequences may be finite or infinite.
    - Finite sequences often are called ***tuples***. A sequence with k elements is a k-***tuple***. Thus (7, 21, 57) is a 3-tuple. A 2-tuple is also called an ***ordered pair***.
- Sets and sequences may appear as elements of other sets and sequences.
    - For example, the ***power set*** of A is the set of all subsets of A. If A is the set {0, 1}, the power set of A is the set { ∅, {0}, {1}, {0, 1} }. The set of all ordered pairs whose elements are 0s and 1s is { (0, 0), (0, 1), (1, 0), (1, 1) }.
- Another example of sequences and tuples may appear as elements in the ***Cartesian product***.
    - If A and B are two sets, the ***Cartesian product*** or ***cross product*** of A and B, written A × B, is the set of all ordered pairs wherein the first element is a member of A and the second element is a member of B.

Sequences and Tuples

Example:

- If A = {1, 2} and B = {x, y, z}, **A × B = { (1, x), (1, y), (1, z), (2, x), (2, y), (2, z) }.**
- We can also take the Cartesian product of k sets, $A_1, A_2, \ldots, A_k$, written $A_1 \times A_2 \times \cdots \times A_k$. It is the set consisting of all k-tuples $(a_1, a_2, \ldots, a_k)$ where $a_i \in A_i$.
- **A × B × A = (1, x, 1), (1, x, 2), (1, y, 1), (1, y, 2), (1, z, 1), (1, z, 2), (2, x, 1), (2, x, 2), (2, y, 1), (2, y, 2), (2, z, 1), (2, z, 2)**
- If we have the Cartesian product of a set with itself, we use the shorthand as-.

$$\overbrace{A \times A \times \cdots \times A}^{k} = A^k.$$

? THE END