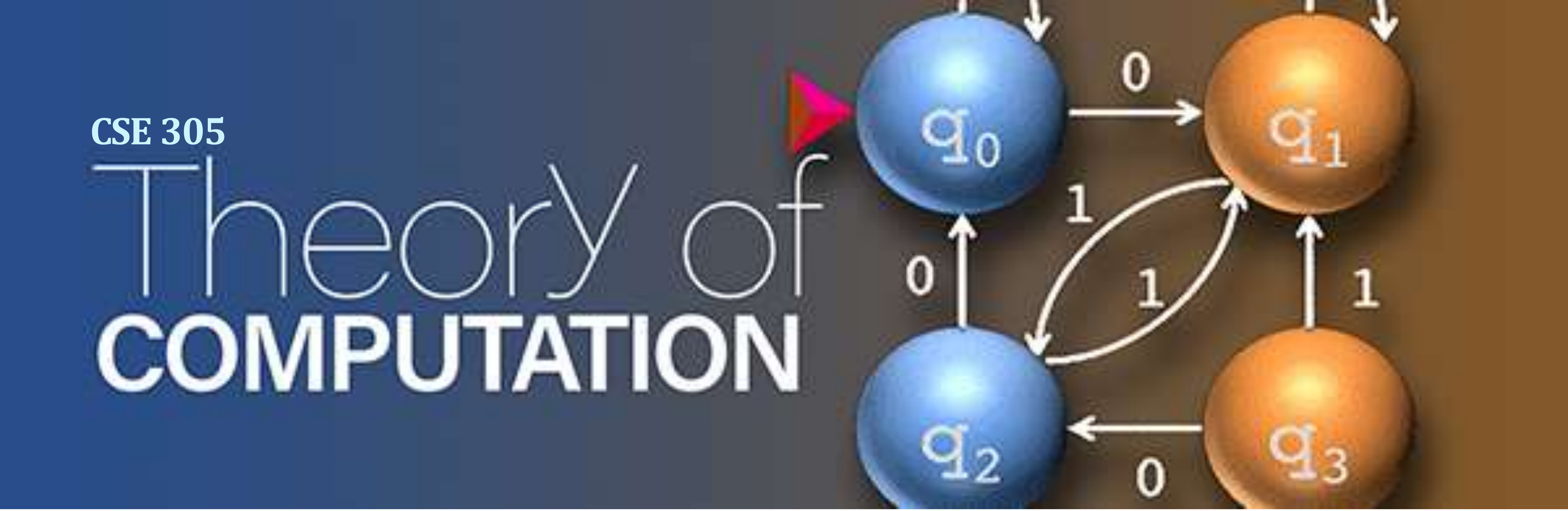

Lecture 6

Mathematical Preliminaries (2)

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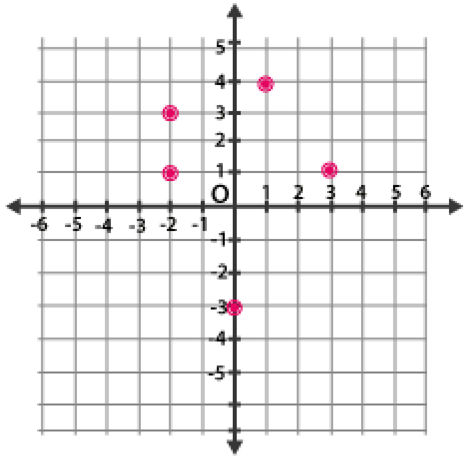
Contents

Mathematical Preliminaries

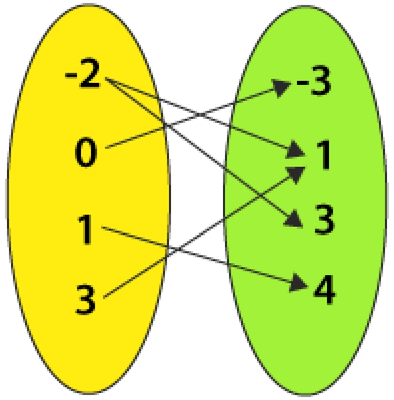
Relations

x	y
-2	1
-2	3
0	-3
1	4
3	1

Relation in table



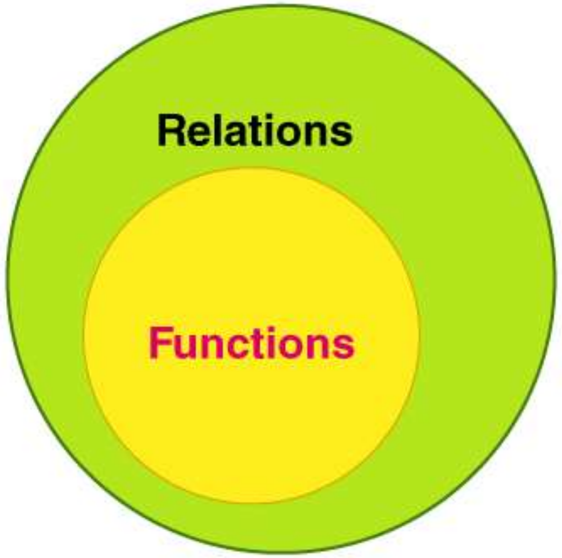
Relation in graph



Relation in mapping diagram

Relations vs. Functions

- **Relations and Functions** are the most important topics in algebra.
- **Relations and functions are the two different words having different meanings mathematically. But, sometimes, we are getting confused about their difference.**
- Let’s understand the difference between both with a simple example:
  - An ordered pair is represented as (INPUT, OUTPUT):
  - The relation shows the relationship between INPUT and OUTPUT.
  - Whereas, a function is a relation which derives one OUTPUT for each given INPUT.
- **Note: All functions are relations, but all relations are NOT functions.**



Relations

- **Relation is a subset of the Cartesian product**. In other words, the relation between the two sets is defined as the collection of the ordered pair, in which the ordered pair is formed by the object from each set.
- Thus, a relation from set A to set B is a subset of the cartesian product set $A \times B$. The subset is made up by describing a relationship between the first element and the second element of elements in $A \times B$.
- R is a relation from A to B, if and only if $R \subseteq A \times B$, e.g., if $(a,b) \in R$, then a is said to be R related to b. The following example clarify the concept of a relation:
  1. If $I = \{x | x \text{ is an integer number}\}$, then we can define a relation R on I, such that $R = \{(a,b) | a,b \in I \text{ and } a<b\}$. It may be written as aRb which implies ‘a is R related to b’.
  2. If $X = \{1,2,3\}$, then we can define a relation R on X, such that $R = \{(a,b) | a,b \in X \text{ and } a=b\}$; so, $R = \{(1,1), (2,2), (3,3)\}$

Relations

Domain and Range of a Relation:

- Consider a relation R from set A to set B is a subset of $A \times B$. Then, the set of all first elements of the ordered pairs of the relation R is called the **domain (D) of R**. Symbolically, we can represent the statement as follows:

$$D = \{a \mid a \in A \text{ and } (a, b) \in R, \text{ for some } b \in B\}$$

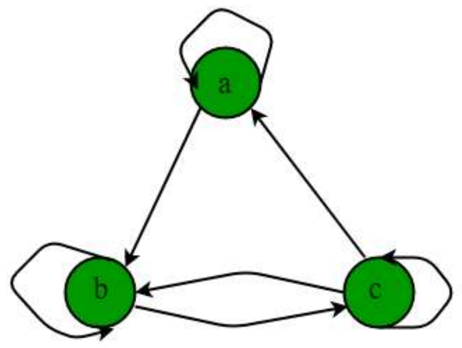
- Similarly, the set of all second elements of the ordered pairs of the relation R is called the **range (E) of R**. Symbolically, we can represent the statement as follows:

$$E = \{b \mid b \in B \text{ and } (a, b) \in R, \text{ for some } a \in A\}$$

- Example: R = {(1,2), (2, -3), (3,5)}. Here, set of all first elements i.e., {1,2,5} is called **domain** while the set of all second elements i.e., {2,-3,5} is called the **range** of the relation.

Relations

The directed graph of relation,
R = {(a,a),(a,b),(b,b),(b,c),(c,c),(c,b),(c,a)}



Representation of a Relation:

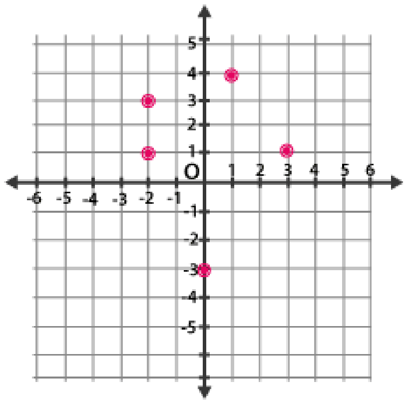
- Various methods are used for the representation of a relation. They are:
  1. Representation using set notation (ordered pairs)
  2. Representation using matrix
  3. Representation using table
  4. Representation using plotting points on XY-graph
  5. Representation using mapping diagram
  6. Representation using directed graph

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

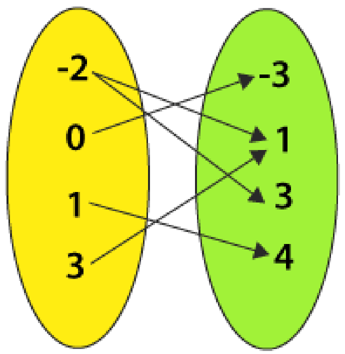
Matrix representation
(Reflexive relation)

x	y
-2	1
-2	3
0	-3
1	4
3	1

Relation in table



Relation in graph



Relation in mapping diagram

Set Notation: R = {(-2,1), (-2,3), (0,-3), (1,4), (3,1)}

Properties and Types of Relations

A relation satisfies a number of properties, such as Reflexivity, Irreflexivity, symmetric, anti-symmetric, Asymmetry, and transitivity. Based on these properties, relations can be classified into different categories. They are:

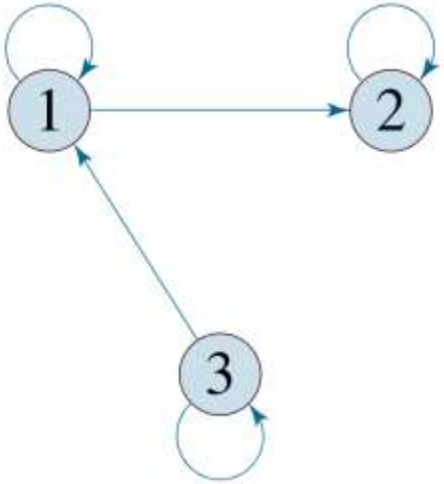
1. Reflexive Relations
2. Symmetric and anti-symmetric Relations
3. Transitive Relations
4. Identity Relations
5. Universal Relations
6. Equivalence Relations
7. Inverse Relations

Properties of Relations

Reflexive Relations:

- A binary relation R is called reflexive if and only if ∀a∈A, aRa. So, a relation R is reflexive if it relates every element of A to itself.
- **Example:** The relation R={(1,1),(1,2), (2,2),(3,3),(3,1)} on the set A = {1, 2, 3}.
- Reflexive relations are always represented by a matrix that has 1 on the main diagonal. The digraph of a reflexive relation has a loop from each node to itself.

$$
\begin{array}{c c}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\end{array}
$$

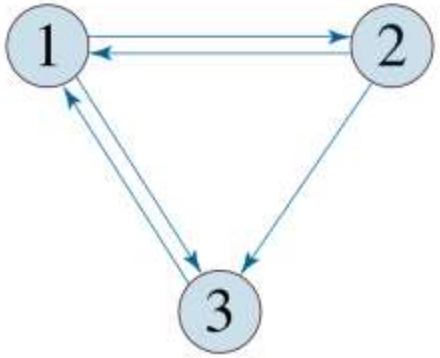


Properties of Relations

Irreflexive Relations:

- A binary relation R on a set A is called irreflexive if aRa does not hold for any a∈A. **This means that there is no element in R which is related to itself.**
- **Example**: The relation R={(1,2),(2,1), (1,3),(2,3),(3,1)} on the set A = {1, 2, 3}.
- The matrix of an irreflexive relation has all 0′s on its main diagonal. The directed graph for the relation has no loops.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$



Properties of Relations

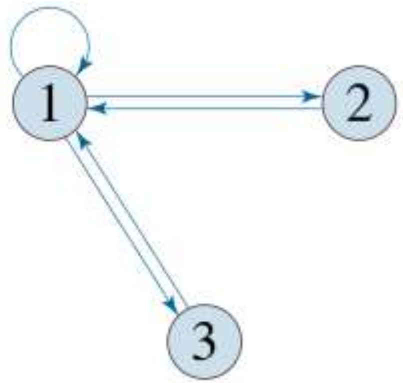
Symmetric Relations:

- A binary relation R on a set A is called symmetric if for all a,b∈A it holds that if aRb then bRa. In other words, the relative order of the components in an ordered pair does not matter - **if a binary relation contains an (a,b) element, it will also include the symmetric element (b,a).**

- **Example:** The relation R={(1,1),(1,2), (2,1),(1,3),(3,1)} on the set A = {1, 2, 3}.

- For a symmetric relation, the logical matrix M is symmetric about the main diagonal. The transpose of the matrix M^T is always equal to the original matrix M. **In a digraph of a symmetric relation, for every edge between distinct nodes, there is an edge in the opposite direction.**

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$



Properties of Relations

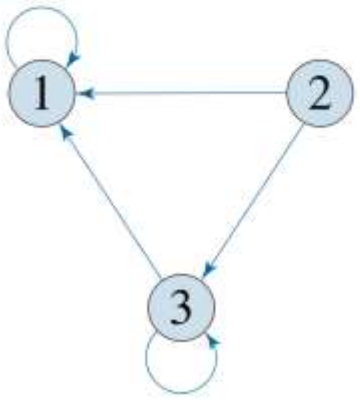
Antisymmetric Relations:

- A binary relation R on a set A is said to be antisymmetric if there is no pair of distinct elements of A each of which is related by R to the other. **So, an antisymmetric relation R can include both ordered pairs (a,b) and (b,a) if and only if a=b.**

- **Example:** The relation R={(1,1),(2,1), (2,3),(3,1),(3,3)} on the set A = {1, 2, 3}.

- In a matrix $M=[a_{ij}]$ representing an antisymmetric relation R, all elements symmetric about the main diagonal are not equal to each other: $a_{ij} \neq a_{ji}$ for $i \neq j$. The digraph of an antisymmetric relation may have loops, however connections between two distinct vertices can only go one way.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

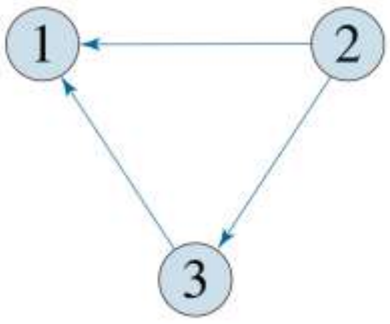


Properties of Relations

Asymmetric Relations:

- An asymmetric binary relation is similar to antisymmetric relation. **The difference is that an asymmetric relation R never has both elements aRb and bRa even if a=b.**
- Every asymmetric relation is also antisymmetric. The converse is not true. If an antisymmetric relation contains an element of kind (a,a), it cannot be asymmetric. Thus, a binary relation R is asymmetric if and only if it is both antisymmetric and irreflexive.
- **Example:** The relation R={(2,1),(2,3),(3,1)} on the set A = {1, 2, 3}.
- The matrix for an asymmetric relation is not symmetric with respect to the main diagonal and contains no diagonal elements. The digraph of an asymmetric relation must have no loops and no edges between distinct vertices in both directions.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

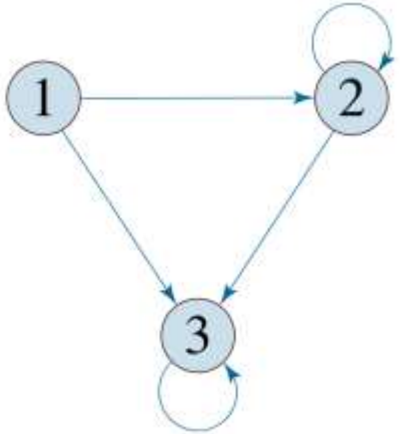


Properties of Relations

Transitive Relations:

- A binary relation R on a set A is called transitive if for all a,b,c∈A it holds that if aRb and bRc, then aRc.
- This condition must hold for all triples a,b,c in the set. **If there exists some triple a,b,c∈A such that (a,b)∈R and (b,c)∈R, but (a,c)∉R, then the relation R is NOT transitive.**
- **Example:** The relation R={(1,2),(1,3), (2,2),(2,3),(3,3)} on the set A = {1, 2, 3}.
- In a matrix M=[aij] of a transitive relation R, for each pair of (i,j)− and (j,k)−entries with value 1 there exists the (i,k)−entry with value 1. The presence of 1′s on the main diagonal does not violate transitivity.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



Properties of Relations

Equivalence Relations:

- **A binary relation on a non-empty set A is said to be an equivalence relation if and only if the relation is**

  reflexive, symmetric, and transitive.

- Two elements a and b related by an equivalent relation are called equivalent elements and generally denoted as $a \sim b$ or $a \equiv b$.
- For an equivalence relation R, you can also see the following notations: $a \sim_R b$, $a \equiv_R b$.
- The equivalence relation is a key mathematical concept that generalizes the notion of equality. It provides a formal way for specifying whether or not two quantities are the same with respect to a given setting or an attribute.

Properties of Relations

Equivalence Relations:

- Example of equivalence relation: **Equality Relation**
- **The equality relation between real numbers or sets, denoted by =, is the canonical example of an equivalence relation. The equality relation R on the set of real numbers is defined by:**

R = { (a,b) | a∈R, b∈R, a=b}.

- R is reflexive since every real number equals itself: a=a.
- R is symmetric: if a=b then b=a.
- The relation R is transitive: if a=b and b=c, then we get-

$$\begin{cases} a = b \\ b = c \end{cases}, \Rightarrow a = b = c, \Rightarrow a = c.$$

Properties of Relations

Universal Relation:

- **Universal relation is a relation on set A when $A \times A \subseteq A \times A$. In other words, universal-relation is the relation R if each element of set A is related to every element of A, i.e $R = A \times A$.**
- It's a full relation as every element of Set A is in Set B.
- Example: A relation on the set $A = \{1,2,3,4,5,6\}$ is given by:

$$R = \{(a,b) \in R : |a - b| \geq 0\}$$

- We observe that $|a - b| \geq 0$ for all $a, b \in A$
  $\Rightarrow (a,b) \in R$ for all $(a,b) \in A \times A$
  $\Rightarrow$ each element of set A is related to every element of set A.
  $\Rightarrow R = A \times A$
  $\Rightarrow$ R is a universal relation on set A.
- **Note :** It is to note here that the void relation and the universal relation on a set A are respectively the smallest and the largest relations on set A. Both the void and universal relation are sometimes called **trivial relations**

Properties of Relations

Identity Relation:

- Let A be a set. Then the relation I = {(a, a) | a ∈ A} on the set A is called **the identity relation on A**. In other words, a relation I on A is called the identity relation, if every element of A is related itself only.

  I = {(a, a) | a ∈ A}.

- **Example:**
  - If A = {1,2,3}, then the relation I = {(1,1), (2,2), (3,3)} is the identity relation on set A. But If we add (1,3) and (3,2) ordered pairs in the set, then it will not be an identity relation.
  - When we throw two dices, the total number of possible ordered pairs is 36; e.g., (1, 1) (1, 2), (1, 3), …, (6, 6). From these, if we consider a relation containing (1, 1), (2, 2), (3, 3) (4, 4) (5, 5) (6, 6) ordered pairs, it is an identity relation.
- What is the difference between an identity relation and a reflexive relation?
  - Any relation 'R' on a set 'A' is said to be reflexive if (a, a) belongs to 'R', for every 'a' belongs to set 'A'. Suppose a relation 'R' defined by R={(1,1), (2,2), (3,3), (1,3), (3,2)} on the set A={1, 2, 3} is a reflexive relation. There can be MANY reflexive relations defined on a set. But, an identity relation on any set is UNIQUE.

Properties of Relations

Inverse Relation:

- **An inverse relation is obtained by interchanging the elements of each ordered pair of the given relation.**
- Let R be a relation from a set A to another set B. Then R is of the form $\{(x, y): x \in A \text{ and } y \in B\}$. The inverse relationship of R is denoted by R^{-1} and its formula is $R^{-1} = \{(y, x) \mid y \in B \text{ and } x \in A\}$. Such that:
  - The first element of each ordered pair of R = the second element of the corresponding ordered pair of R^{-1} and
  - The second element of each ordered pair of R = the first element of the corresponding ordered pair of R^{-1}.

- **Example:** Consider a relation R on two sets $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5\}$.

  If $R = \{(a, 2), (b, 4), (c, 1)\} \Leftrightarrow R^{-1} = \{(2, a), (4, b), (1, c)\}$.

  Here the domain is the range R^{-1} and vice versa.

THE END