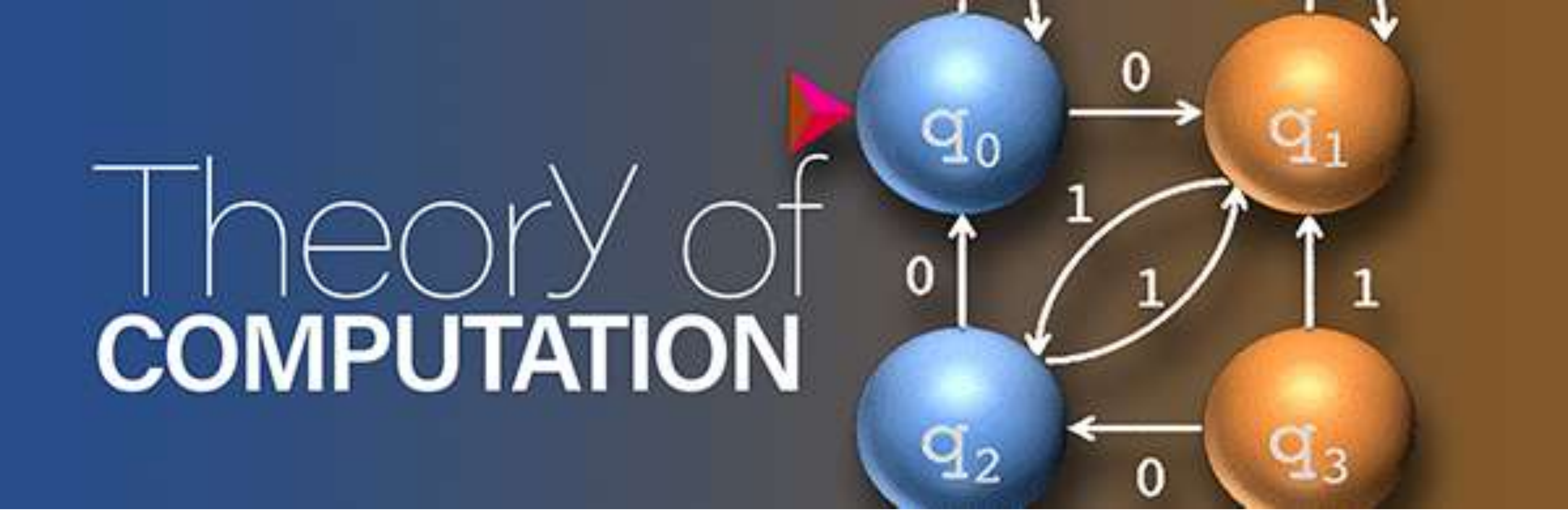

Lecture 7

Mathematical Preliminaries (3)

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Contents

Mathematical Preliminaries

Closures of Relations

- **Let R be a binary relation on a set A. The relation R may or may not have some property P, such as reflexivity, symmetry, or transitivity.**
- Suppose, for example, that R is not reflexive. If so, we could add ordered pairs to this relation to make it reflexive. **The smallest reflexive relation R^+ that includes R is called the reflexive closure of R.**

- In general, if a relation R^+ with property P contains R such that:
    - R^+ is a subset of every relation with property P containing R, then R^+ is a closure of R with respect to property P.

Closures of Relations

- There are many ways to denote closures of relations.
- Besides the common notations like R+, **the reflexive closure** of a relation R may be denoted by

  R_r, Rr, R^+_r, r(R), clref(R), refc(R), etc.

- For **the symmetric closure**, the following notations can be used:

  R_s, Rs, R^+_s , s(R), clsym(R), symc(R), etc.

- Respectively**, the transitive closure** is denoted by

  R_t, Rt, R^+_t , t(R), cltrn(R), tr(R), etc.

- Commonly used notations: R^+, r(R), s(R), t(R).

Closures of Relations

Reflexive Closure:

- The reflexive closure of a binary relation R on a set A is defined as the smallest reflexive relation r(R) on A that contains R. The smallest relation means that it has the fewest number of ordered pairs. The reflexive closure r(R) is obtained by adding the elements (a,a) to the original relation R for all a ∈ A. Formally, we can write:

  $r(R)=R \cup I$,

  where I is the identity relation, which is given by

  $I=\{(a, a) \mid \forall a \in A\}$.

- **Example:** Consider the relation R={(1,2), (2,4), (3,3), (4,2)} on the set A={1, 2, 3, 4}. R is not reflexive. To make it reflexive, we add all missing diagonal elements:

$$r(R) = R \cup I = \{(1,2), (2,4), (3,3), (4,2)\} \cup \{(1,1), (2,2), (3,3), (4,4)\}$$

$$= \{(1,1), (1,2), (2,2), (2,4), (3,3), (4,2), (4,4)\}.$$

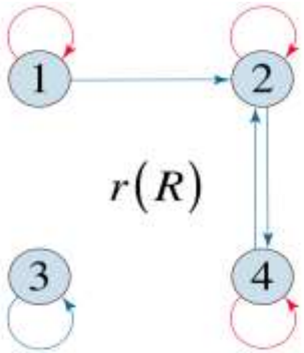
Closures of Relations

Reflexive Closure:

- **From the previous example,** r(R) = R∪I = {(1,2), (2,4), (3,3), (4,2)} ∪ {(1,1), (2,2), (3,3), (4,4)} = {(1,1), (1,2), (2,2), (2,4), (3,3), (4,2), (4,4)}.
- The matrix of the reflexive closure of R is given by:

$$M_{r(R)} = M_R + M_I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- The digraph of the reflexive closure r(R) is obtained from the digraph of the original relation R by adding missing self-loops to all vertices.



Closures of Relations

Symmetric Closure:

- The symmetric closure of a relation R on a set A is defined as the smallest symmetric relation s(R) on A that contains R. The symmetric closure s(R) is obtained by adding the elements (b, a) to the relation R for each pair (a, b) ∈ R. In terms of relation operations, the symmetric closure s(R) is:

  $s(R) = R \cup R^{-1} = R \cup R^{T}$, where $R^{-1} = R^{T}$ denotes the inverse of R (also called the converse or transpose relation).

- **Example:** Let R = {(1,2), (1,3), (2,2), (2,4), (4,3)} be a binary relation on the set A={1,2,3,4}. The relation R is not symmetric. It contains 4 non-reflexive elements: (1,2), (1,3), (2,4), and (4,3), which do not have a reverse pair. So, to make R symmetric, we need to add the following missing reverse elements: (2,1), (3,1), (4,2), and (3,4):

$$s(R) = \{(1,2), (1,3), (2,2), (2,4), (4,3)\} \cup \{(2,1), (3,1), (4,2), (3,4)\}$$

$$= \{(1,2), (1,3), (2,1), (2,2), (2,4), (3,1), (3,4), (4,2), (4,3)\}.$$

Closures of Relations

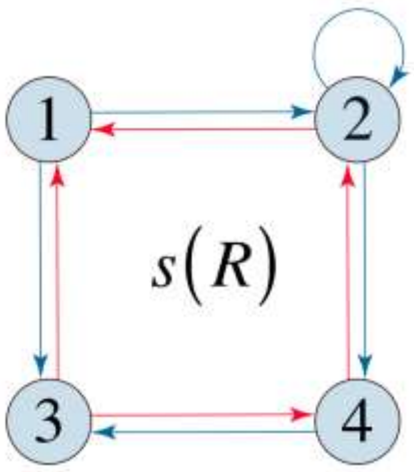
Symmetric Closure:

- **From the example,** s(R) = {(1,2), (1,3), (2,2), (2,4), (4,3)} ∪ {(2,1), (3,1), (4,2), (3,4)}

  = {(1,2), (1,3), (2,1), (2,2), (2,4), (3,1), (3,4), (4,2), (4,3)}.

- The matrix of the symmetric closure of R is given by summing of the matrices M_R and $M_{R\text{-}1}$:

$$M_{s(R)} = M_R + M_{R^{-1}} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



- The digraph of the symmetric closure s(R) is obtained from the digraph of the original relation R by adding the edge in the reverse direction (if none already exists) for each edge in the digraph for R:

Closures of Relations

Transitive Closure:

- The transitive closure of a binary relation R on a set A is the smallest transitive relation t(R) on A containing R. The transitive closure is more complex than the reflexive or symmetric closures.

- **To describe how to construct a transitive closure, we need to introduce two new concepts - the paths and the connectivity relation.**

- **Paths:** Suppose that R is a relation on a set A. Consider two elements a∈A, b∈A. A path from a to b of length n is a sequence of ordered pairs

$(a, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, b)$, in the relation R, where n is a nonnegative integer.

Closures of Relations

Transitive Closure:

- **Example for paths:** Let R = {(1,2), (2,4), (4,3)} be a relation on set A={1,2,3,4}. All the pairs (1,2), (2,4), (4,3) are the paths of length n=1. Besides that, R has the paths of length n=2:

  (1,2), (2,4) and (2,4), (4,3).

- It can also be seen that the relation R itself is a path of length n=3.

- **Theorem:** If R is a relation on a set A and a∈A, b∈A, then there is a path of length n from a to b if and only if (a,b)∈Rn for every positive integer n.

Closures of Relations

Transitive Closure:

- **Connectivity Relation:** The connectivity relation of R, denoted R^*, consists of all ordered pairs (a,b) such that there is a path (of any length) in R from a to b.

- The connectivity relation R* is the union of all the sets R^n:

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

- If the relation R is defined on a finite set A with the cardinality $|A| = n$, then the connectivity relation is given by:

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n.$$

- It is clear that if $R_{i-1} = R_i$ where $i \leq n$, we can stop the computation process since the higher powers of R will not change the union operation.

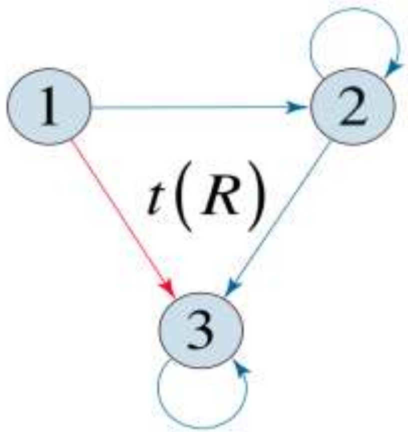
Closures of Relations

Finding Transitive Closure:

- The transitive closure t(R) of a relation R is equal to its connectivity relation R∗.
- Consider the relation R = {(1,2), (2,2), (2,3), (3,3)} on the set A={1,2,3}. R is not transitive since we have (1,2) ∈R, (2,3)∈R, but (1,3)∉R. So we need to add (1, 3) to make R transitive:

  t(R) = R∪{(1,3)} = {(1,2), (2,2), (2,3), (3,3)} ∪ {(1,3)} = {(1,2), (1,3), (2,2), (2,3), (3,3)}.

- The digraph of a transitive closure contains all edges from a to b if there is a directed path from a to b. In this example, the transitive closure t(R) is represented by the following digraph:



Closures of Relations

Finding Transitive Closure:

- We can also find the transitive closure of R in matrix form. The original relation R is defined by the matrix:

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- The connectivity relation R∗ is determined by the expression: $R^* = R \cup R^2 \cup R^3$.

- Calculate the matrix of the composition R^2:

$$M_{R^2} = M_R \times M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+1+0 & 0+1+0 \\ 0+0+0 & 0+1+0 & 0+1+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Closures of Relations

Finding Transitive Closure:

- Similarly, we calculate the matrix of the composition R^3:

$$M_{R^3} = M_{R^2} \times M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+1+0 & 0+1+1 \\ 0+0+0 & 0+1+0 & 0+1+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- As it can be seen, $M_{R^2} = M_{R^3}$. Hence, the connectivity relation R^* can be found by the formula:

  $R^* = R \cup R^2$.

- Using matrix representation, we have: $$M_{R^*} = M_R + M_{R^2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- This matrix addition is performed based on the Boolean arithmetic rules.

Closures of Relations

Finding Transitive Closure:

- The computed matrix of transitive closure R*: $M_{R^*} = M_R + M_{R^2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

- Converting this into roster form, we obtain the relation as:

  t(R) = R* = {(1,2), (1,3), (2,2), (2,3), (3,3)}.

- The algorithm involving calculation of the connectivity relation has the running time proportional to O(n^4). There are faster methods of finding transitive closures. For example, the **Warshall algorithm** allows to compute the transitive closure of a relation with the rate of O(n^3).

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