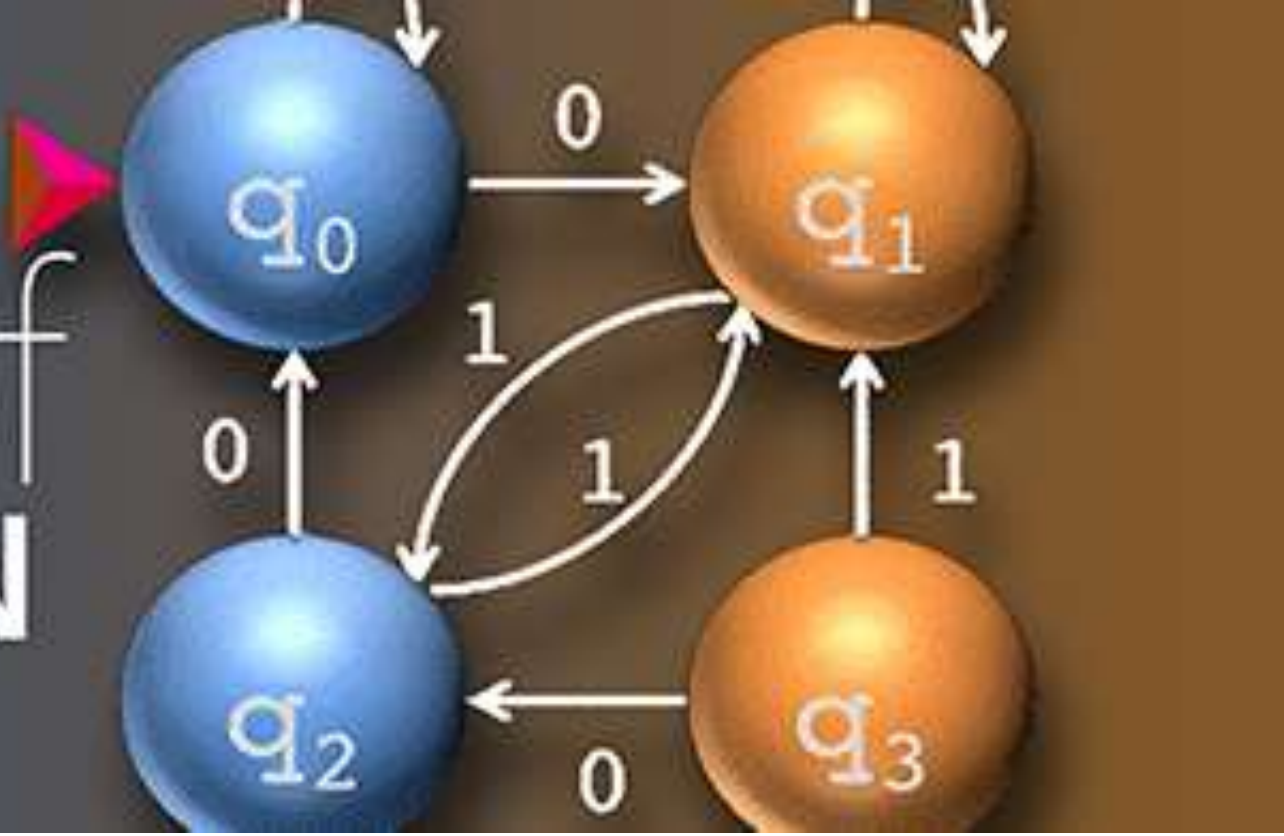
CSE 305

Theory of COMPUTATION



Lecture 10

Mathematical Preliminaries (6)

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Mathematical Preliminaries

- Set Theory
- Sequences and Tuples
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Mathematical Logic

Mathematical Logic

- **Logic is concerned with the truth of statements about the world. Generally each statement is either TRUE or FALSE.**
- **Logic includes : *Syntax* , *Semantics and Inference Procedure*.**
  - **Syntax :** Specifies the *symbols* in the language about how they can be combined to form sentences. The facts about the world are represented as sentences in logic.
  - **Semantic :** Specifies how to assign a truth value to a sentence based on its *meaning* in the world. It Specifies what facts a sentence refers to. A fact is a claim about the world, and it may be *TRUE* or *FALSE*.
  - **Inference Procedure :** Specifies *methods* for computing new sentences from an existing sentences.
- Logics are of different types, such as Propositional logic, Predicate logic, Temporal logic, Modal logic, Description logic etc. **Propositional logic and Predicate logic** are fundamental to all logic.

Propositional Logic

- **Propositional logic (PL) is the simplest form of logic where all the statements are made by propositions.**
- A proposition is a declarative statement which is either true or false, but it cannot be both. It is a technique of knowledge representation in logical and mathematical form.

- **Example:**

  a) It is Sunday.

  b) The Sun rises from West (False proposition)

  c) 3+3= 7 (False proposition)

  d) 5 is a prime number.

Propositional Logic

- **Propositional logic is also called Boolean logic as it works on 0 and 1.** We use symbolic variables to represent the logic, and we can use any symbol for a representing a proposition, such A, B, C, P, Q, R, etc.
- **Propositional logic consists of an object, relations or function, and logical connectives.** These connectives are also called logical operators. **The propositions and connectives are the basic elements of the propositional logic**.
- **Syntax of propositional logic:** The syntax of propositional logic defines the allowable sentences for the knowledge representation. There are two types of Propositions:
  - **Atomic Propositions**
  - **Compound propositions**

Propositional Logic

- **Atomic Proposition:** Atomic propositions are the simple propositions. It consists of a single proposition symbol. These are the sentences which must be either true or false.
- **Example:**

a) 2+2 is 4, it is an atomic proposition as it is a **true** fact.

b) "The Sun is cold" is also a proposition as it is a **false** fact.

- **Compound proposition:** Compound propositions are constructed by combining simpler or atomic propositions, using parenthesis and logical connectives.
- **Example:**

a) "It is raining today, and street is wet."

b) "Ankit is a doctor, and his clinic is in Mumbai."

Propositional Logic

- **Logical Connectives:** Logical connectives are used to connect two simpler propositions or representing a sentence logically. We can create compound propositions with the help of logical connectives.
- **There are mainly five connectives, such as negation, conjunction, disjunction, implication, and bidirectional.**

1. Negation: A sentence such as ¬ P is called negation of P. A literal can be either Positive literal or negative literal.

2. Conjunction: A sentence which has **∧** connective such as, **P ∧ Q** is called a conjunction. For **example:** Robin is intelligent and hardworking. It can be written as-

P= Robin is intelligent, Q= Robin is hardworking. We can write it as P∧ Q.

3. Disjunction: A sentence which has ∨ connective, such as **P ∨ Q**. is called disjunction, where P and Q are the propositions. For **example: "John is a doctor or Engineer"**, Here P= John is Doctor. Q= John is Engineer, so we can write it as **P ∨ Q**.

Propositional Logic

4. Implication: A sentence such as P → Q, is called an implication. Implications are also known as if-then rules. It can be represented as-

If it is raining, then the street is wet.

Let P= It is raining, and Q= Street is wet, so it is represented as P → Q

5. Biconditional: A sentence such as **P⇔ Q is a Biconditional sentence, example If I am breathing, then I am alive. It can be represented as-**

P= I am breathing, Q= I am alive, it can be represented as P ⇔ Q.

- **Following is the summarized table for Propositional Logic Connectives:**

Connective symbols	Word	Technical term	Example
∧	AND	Conjunction	A ∧ B
∨	OR	Disjunction	A ∨ B
→	Implies	Implication	A → B
⇔	If and only if	Biconditional	A⇔ B
¬ or ~	Not	Negation	¬ A or ¬ B

Propositional Logic

- **Truth Table:**
  - In propositional logic, we need to know the truth values of propositions in all possible scenarios.
  - We can combine all the possible combination with logical connectives, and the representation of these combinations in a tabular format is called **Truth table**.

p	q	¬p	¬q	p ∧ q	p ∨ q	p→q	p ↔ q	q→p
T	T	F	F	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	F	F	T	T	F	F
F	F	T	T	F	F	T	T	T

Propositional Logic

Precedence of connectives:

- Just like arithmetic operators, there is a precedence order for propositional connectors or logical operators. This order should be followed while evaluating a propositional problem.
- Following is the list of the precedence order for operators:

Precedence	Operators
First Precedence	Parenthesis
Second Precedence	Negation
Third Precedence	Conjunction(AND)
Fourth Precedence	Disjunction(OR)
Fifth Precedence	Implication
Six Precedence	Biconditional

Propositional Logic

- **Logical equivalence:** Logical equivalence is one of the features of propositional logic. Two propositions are said to be logically equivalent if and only if the columns in the truth table are identical to each other.
- Let's take two propositions A and B, so for logical equivalence, we can write it as A⇔B. In below truth table we can see that column for ¬A∨ B and A→B, are identical hence A is Equivalent to B

A	B	¬A	¬A∨ B	A→B
T	T	F	**T**	**T**
T	F	F	**F**	**F**
F	T	T	**T**	**T**
F	F	T	**T**	**T**

Propositional Logic

- **Properties of Operators:**

Commutativity:

$P \wedge Q = Q \wedge P$; $P \vee Q = Q \vee P$.

Associativity:

$(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$; $(P \vee Q) \vee R = P \vee (Q \vee R)$

Identity element:

$P \wedge True = P$; $P \vee True = True$.

Distributive:

$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$; $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$.

DE Morgan's Law:

$\neg (P \wedge Q) = (\neg P) \vee (\neg Q)$; $\neg (P \vee Q) = (\neg P) \wedge (\neg Q)$.

Double-negation elimination:

$\neg (\neg P) = P$.

Propositional Logic

Limitations of Propositional logic:

- We cannot represent relations like ALL, some, or none with propositional logic. Example:
    - **All the girls are intelligent.**
    - **Some apples are sweet.**
- Propositional logic has limited expressive power.
- In propositional logic, we cannot describe statements in terms of their properties or logical relationships.

Predicate Logic

- The ***propositional logic, is not powerful*** enough for all types of assertions. For example, the assertion *"x > 1"*, where ***x*** is a variable, is not a proposition because it is neither true nor false unless value of ***x*** is defined.
  - For *x > 1* to be a proposition,
    – either we substitute a specific number for ***x*** ;
    – or change it to something like *"There is a number x for which x > 1 holds"*;
    – or *"For every number x, x > 1 holds"*.
  - Consider another example :
    "All men are mortal.
    Socrates is a man.
    Then Socrates is mortal" ,
  - These cannot be expressed in propositional logic as a finite and logically valid formula.

Predicate Logic

- **Predicate Logic (also known as Predicate calculus)** expands on propositional logic by introducing variables, usually denoted by x, y, z, or other lowercase letters.
- It also introduces sentences containing variables, called predicates, usually denoted by an uppercase letter followed by a list of variables, such as P(x) or Q(y,z).

- **A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.**
- The following are some examples of predicates −

  Let E(x, y) denote "x = y"

  Let X(a, b, c) denote "a + b + c = 0"

  Let M(x, y) denote "x is married to y"

Logical Quantifiers

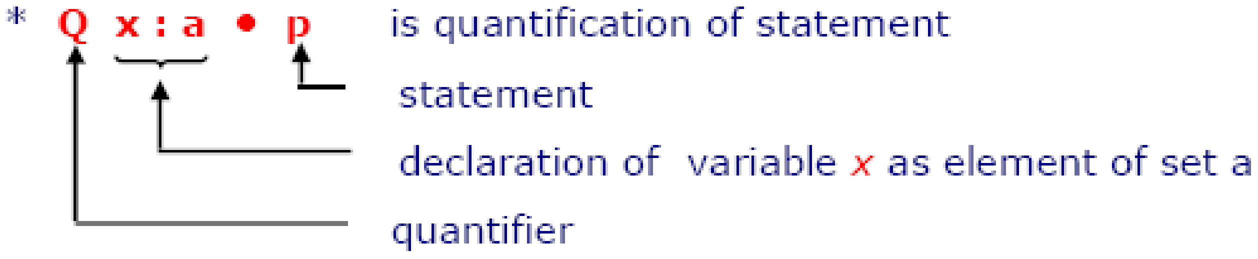
- **Generally, a predicate with variables can be made a proposition by applying one of the following two operations to each of its variables :**

  1. Assign a value to the variable; e.g., x > 1, if 3 is assigned to x becomes 3 > 1, and it then becomes a true statement, hence a proposition.

  2. Quantify the variable using a quantifier on formulas of predicate logic, such as x > 1 or P(x), by using Quantifiers on variables.

- **Apply Quantifiers on Variables:**

‡ **Statement** **p** is a statement about **x**



* Quantifiers are two types :

universal quantifiers , denoted by symbol ∀ and

existential quantifiers , denoted by symbol ∃

Logical Quantifiers

- **The Universal Quantifier:**
- **The "all" form.** The universal quantifier is frequently encountered in the following context:

  $\forall x(P(x) \Rightarrow Q(x))$,

  which may be read, "All x satisfying P(x) also satisfy Q(x).
- A sentence $\forall x P(x)$ is true if and only if P(x) is true no matter what value (from the universe of discourse) is substituted for x.

‡ In propositional form it is written as : $\forall x \; P(x)$

* read " *for all* *x*, *P(x) holds* "
" *for each* *x*, *P(x) holds* " *or*
" *for every* *x*, *P(x) holds* "

* where *P(x)* is predicate,
$\forall x$ means all the objects *x* in the universe
P(x) is true for every object *x* in the universe

Example:

$\forall x \, (x^2 \geq 0)$, i.e., "the square of any number is not negative."

Logical Quantifiers

- **The Existential Quantifier:**

- **The "some" form.** The existential quantifier is frequently encountered in the following context:

  $\exists x\ (P(x) \wedge Q(x))$,

  which may be read, "Some x satisfying P(x) also satisfies Q(x)".'

- A sentence ∃xP(x)∃xP(x) is true if and only if there is at least one value of xx (from the universe of discourse) that makes P(x)P(x) true.

- **Example:**

  ∃x(x is a prime number ∧ x is even), i.e., "some prime number is even".'

Logical Quantifiers

- **The Existential Quantifier:**

‡ In propositional form it is written as : $\exists\ x\ P(x)$

* read " *there exists an x such that P(x)* " or

" *there exists at least one x such that P(x)* "

* Where $P(x)$ is predicate

$\exists x$ means at least one object x in the universe

$P(x)$ is true for least one object x in the universe

‡ Example : English language to Propositional form

* " *Someone loves you* "

$\exists$ *x : Someone • x loves you*

* $x\ P(x)$

where $P(x)$ is predicate tells : ' *x loves you* '

x is variable for object ' *someone* ' that populate universe of discourse

? THE END